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A Yosida-Hewitt decomposition for totally monotone games on locally compact σ-compact topological spaces

Yann Rébillé *

Abstract

We prove for totally monotone games defined on the set of Borel sets of a locally compact σ-compact topological space a similar decomposition theorem to the famous Yosida-Hewitt's one for finitely additive measures. This way any totally monotone decomposes into a continuous part and a pathological part which vanishes on the compact subsets. We obtain as corollaries some decompositions for finitely additive set functions and for minitive set functions.

Résumé

Nous obtenons une décomposition à la Yosida-Hewitt pour les jeux totalement monotones définis sur un espace topologique localement compact et σ-compact. Ainsi tout jeu totalement monotone se décompose en une partie continue et une partie pathologique s'annulant sur les compacts. Nous obtenons en corollaires des décompositions pour les fonctions d'ensemble additives et pour les fonctions d'ensemble minitives.

Keywords: Choquet's integral representation theorem, Yosida-Hewitt decomposition, totally monotone games.

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1 Introduction

The interest in obtaining a decomposition à la Yosida–Hewitt ([14]) for additive measures an a measurable space has now been recognized for more than 50 years. Among other merits this decomposition theorem allows to separate a continuous additive component of a finitely additive measure from its "pathological" purely non-continuous part. Since Choquet's ([4]) seminal contribution to the study of capacities in potential theory, there has been a growing interest on non-additive set functions in various scientific fields such as artificial intelligence, game theory or decision theory. Of particular interest is the theory of evidence ([12]), the transferable belief model ([13]) based on totally monotone set functions (conjugate of infinite alternated capacities), or the theory of possibilities ([6]) and fuzzy sets ([15]) based on maxitive set functions. A natural question arises of knowing if such a decomposition à la Yosida–Hewitt can be maintained for these set functions (see [9]). A first positive attempt has been achieved in [3] where the study was devoted to countable measurable spaces. Our aim here is to pursue in this direction, and to consider the topological case.
Section 2 introduces the needed preliminary material, including the Choquet's integral representation theorem. In section 3 we state and prove for totally monotone games on a locally compact σ-compact topological space a natural generalization of the Yosida-Hewitt decomposition for totally monotone games on $\mathcal{P}(\mathbb{N})$ ([3]). As a byproduct we retrieve some decompositions for finitely additive measures and for minitive measures. Section 4 extend this results for totally monotone comonotone additive functionals on the set of non-negative bounded Borel functions.

2 Definitions, notations and preliminary results

Let (Ω, τ) be a Hausdorff topological space and $Bor(\tau)$ the σ-algebra of Borel sets. The set of compact subsets will be denoted by $\mathcal{K}$ and $cl(\tau) = \{A : A^c \in \tau\}$ denotes the set of closed sets. For $A \subset \Omega$, A^o and $\overline{A}$ denote respectively the interior and the closure of A.

Our hypothesis from now is that Ω is a locally compact σ-compact topological space [1] (e.g. $\mathbb{R}^n$). Or equivalently (see [1]) there exists a sequence $\{K_n\}_n \subset \mathcal{K}$ such that $\cup_n K_n = \Omega$ satisfying $K_n \subset K^o_{n+1}$.

A real valued set function v on $Bor(\tau)$ is said to be a *game* if $v(\emptyset) = 0$.
A game is said to be *monotone* if $\forall A, B \in Bor(\tau), A \supset B \Rightarrow v(A) \geq v(B)$. Hence v is *non-negative* i.e. $v \geq 0$.
A non-negative game v is said to be a *(finitely additive) measure* if $\forall A, B \in Bor(\tau), A \cap B = \emptyset,\ v(A \cup B) = v(A) + v(B)$. Furthermore, if $v(\Omega) = 1$, v is a *probability*.

[1] For our interest, Ω will not be compact.

Given an integer $K \geq 2$ a game v is said to be *monotone of order K* if $\forall A_1, \ldots, A_K \in Bor(\tau)$,

$$v(\cup_{k=1}^{K} A_k) \geq \sum_{\{I:\emptyset \neq I \subset \{1,\ldots,K\}\}} (-1)^{|I|+1} v(\cap_{k\in I} A_k)$$

where $|I|$ denotes the cardinal of I.
If a game v is monotone and monotone of order K for all $K \geq 2$, v is said to be totally monotone. Furthermore, if $v(\Omega) = 1$, v is a *belief function* ([12]). In particular a measure is a totally monotone game.

An other important subclass of totally monotone games is constituted of minitive games ([9]). A game v is said to be minitive if

$$\forall A, B \in Bor(\tau), v(A \cap B) = \min\{v(A), v(B)\}$$

Furthermore, if $v(\Omega) = 1$, v is a *necessity measure* ([6]).

A game v is said to be τ-continuous if for all non-decreasing sequences $\{A_n\}_n \subset Bor(\tau)$,

$$A_n \subset A_{n+1}^o, \cup_n A_n \in cl(\tau) \Rightarrow \lim_\infty v(A_n) = v(\cup_n A_n)$$

and for all non-increasing sequences $\{A_n\}_n \subset Bor(\tau)$,

$$A_n \supset \overline{A_{n+1}}, \cap_n A_n \in \tau \Rightarrow \lim_\infty v(A_n) = v(\cap_n A_n)$$

By definition, τ-continuity takes place for clopen sets i.e. on $\tau \cap cl(\tau)$ [2].
If v is monotone and monotone of order 2, then v is τ-continuous if and only if v is *continuous at* Ω ([10], see Appendix) i.e. for all non-decreasing sequences $\{A_n\}_n \subset Bor(\tau)$,

$$A_n \subset A_{n+1}^o, \cup_n A_n = \Omega \Rightarrow \lim_\infty v(A_n) = v(\Omega)$$

Typical example of τ-continuous totally monotone games are given by *unanimity games* [3]: for $K \in \mathcal{K}, K \neq \emptyset$ let u_K be the unanimity game on $Bor(\tau)$ defined by

$$\forall A \in Bor(\tau), u_K(A) = \begin{cases} 1, & \text{if } K \subset A \\ 0, & \text{otherwise.} \end{cases}$$

A totally monotone game v is said to be *pure* if $\forall$ ν τ-continuous totally monotone game, $0 \leq \nu \leq v \Rightarrow \nu = 0$.
Similarly to the countable case we obtain a characterization of purity in terms of compact subsets,

Lemma 1 *Let v be a totally monotone game on $Bor(\tau)$ then v is pure if and only if $v_{|\mathcal{K}} = 0$.*

[2] τ-continuity is a weaker property than σ-continuity which requires that convergence should hold for any monotone sequence $\{A_n\} \subset Bor(\tau)$.

[3] Unanimity game are σ-continuous if and only if K is finite.

Proof : (if) Let ν be a τ-continuous totally monotone game with $0 \leq \nu \leq v$. Since $v_{|\mathcal{K}} = 0$, for all $n \in \mathbb{N}$ we have, $0 \leq \nu(K_n) \leq v(K_n) = 0$. Now τ-continuity of ν entails $\nu(\Omega) = 0$, thus $\nu = 0$ by monotonicity.
(only if) Assume there exists $K \in \mathcal{K}$ such that $v(K) > 0$. Take $\nu_K = v(K)u_K$, then $0 \leq \nu_K \leq v$ and $\nu_K \neq 0$. It remains to prove that ν_K is τ-continuous at Ω. Let $\{A_n\}_n \subset Bor(\tau)$ be a non-decreasing sequence with $A_n \subset A^o_{n+1}, \cup_n A_n = \Omega$. Since $K \in \mathcal{K}$ there exists n_K such that $K \subset A_{n_K}$, so $\nu_K(A_{n_K}) = v(K) = \nu_K(\Omega)$. □

In order to obtain a proof of our decomposition theorem for totally monotone games we shall use as in [3] the following version of Choquet's integral representation theorem (see p.268 in [7]). For this we recall the setting.

Let K be a nonempty compact convex subset of a locally convex Hausdorff vector space E. Denote by $A(K)$ the space of affine continuous functions on K. A function φ is said to be *affine* if $\forall x, y \in K, \forall \lambda \in [0,1]$,

$$\varphi(\lambda x + (1-\lambda)y) = \lambda\varphi(x) + (1-\lambda)\varphi(y).$$

A point $x \in K$ is said to be an *extreme point* of K if $\forall\ y, z \in K, \forall \lambda \in]0,1[, x = \lambda y + (1-\lambda)z \Rightarrow x = y = z$. Denote $ex(K)$ the set of extreme points of K.

Theorem: (Choquet) *For every $x \in K$, there is a σ-additive probability m_x on $ex(K)$ (with respect to the smallest σ-algebra making all elements of $A(K)_{|ex(K)}$ measurable) such that for all $h \in A(K)$:*

$$h(x) = \int_{ex(K)} h_{|ex(K)} dm_x$$

A family $\mathcal{F}$ of subsets of $Bor(\tau)$ is said to be a *filter* (see e.g. [1] p. 31) if,
(i) $\emptyset \notin \mathcal{F}, \Omega \in \mathcal{F}$,
(ii) $\forall A, B \in Bor(\tau), [A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}]$,
(iii) $\forall A, B \in Bor(\tau), [B \supset A, A \in \mathcal{F} \Rightarrow B \in \mathcal{F}]$.

Let $\mathcal{F}$ be a filter of $Bor(\tau)$, define the *filter game* $u_{\mathcal{F}}$ [4],

$$u_{\mathcal{F}}(B) = \begin{cases} 1, & \text{if } B \in \mathcal{F} \\ 0, & \text{otherwise.} \end{cases}$$

It turns out (see [2], [4] pages 260-261), that for K the set of belief functions on $Bor(\tau)$, one obtains that the set $ex(K)$ of extreme points of K consists of the filter games, in other words of the $\{0,1\}$-valued belief functions, those that take only the values zero and one.
Finally we prove :

Lemma 2 *Let $u_{\mathcal{F}}$ be a filter game on Ω, then the following assertions are equivalent,*
(i) *$u_{\mathcal{F}}$ is τ-continuous,*
(ii) *$u_{\mathcal{F}}$ is not pure.*

[4]Filter games coincide with $\{0,1\}$-valued minitive games.

Proof : $(i) \Rightarrow (ii)$. is immediate.
$(ii) \Rightarrow (i)$. If $u_{\mathcal{F}}$ is not pure, from Lemma 1 there exists $K \in \mathcal{K}, \neq \emptyset$ such that $u_{\mathcal{F}}(K) > 0$, thus $u_{\mathcal{F}} \geq u_K$. But since u_K is τ-continuous at Ω, $u_{\mathcal{F}}$ is also τ-continuous at Ω, thus τ-continuous. □

As an example of pure filter game we can consider $u_{co\mathcal{K}}$, where $co\mathcal{K} = \{A : A^c \in \mathcal{K}\}$ is the Fréchet filter of cocompact sets.

3 Decomposition of totally monotone games

Theorem 1 *Let v be a totally monotone game then there exists a unique pair of totally monotone games (v_c, v_p) with v_c τ-continuous and v_p pure such that $v = v_c + v_p$.*

Proof : If $v(\Omega) = 0$ it is immediate. Without loss of generality we will assume that v is a belief function i.e. $v(\Omega) = 1$.
Denote by E the linear space of games on $Bor(\tau)$ where $\forall v, w \in E, \forall \lambda \in \mathbb{R}, \forall A \in Bor(\tau), (v + w)(A) = v(A) + w(A)$ and $(\lambda.v)(A) = \lambda v(A)$. Let us endow E with the topology of point-wise convergence. Under this topology the vector space E becomes a locally convex and Hausdorff topological vector space (see [8]). Denote K the set of belief functions on $Bor(\tau)$. From [2], [3], K is a compact convex subset of E.

Recall that the set $ex(K)$ of extreme points of K consists of the filter games. Let us denote $ex\tau(K)$ (resp. $ex\pi(K)$) the set of τ-continuous (resp. pure) extremal elements of K . Lemma 2 asserts that the set of pure extremal elements of K is the complement of $ex\tau(K)$ within $ex(K)$.

Denote Σ_K the smallest σ-algebra of subsets of $ex(K)$ making all elements of $A(K)_{|ex(K)}$ measurable.
For $A \in Bor(\tau)$, denote $\tilde{A} = \{u_{\mathcal{F}} : u_{\mathcal{F}}(A) = 1\}$ as in [2].
Consider the application $h_A : E \longrightarrow \mathbb{R} : v \mapsto v(A)$, where $A \in Bor(\tau)$. h_A is affine and continuous by construction, hence $h_{A|ex(K)}$ is Σ_K-measurable. So $\{h_{A|ex(K)} \geq 1\} = \tilde{A} \in \Sigma_K$.
From Lemma 1, we obtain that the set of pure extremal elements of K consists of the filter games which vanish on the compact sets. Since for all $K \in \mathcal{K}$ there exists n such that $K \subset K_n$, we have that

$$ex\pi(K) = \cap_n\{u_{\mathcal{F}} : u_{\mathcal{F}}(K_n) = 0\} = \cap_n(\tilde{K_n})^c \text{ and } ex\tau(K) = \cup_n \tilde{K_n}.$$

Now Σ_K being a σ-algebra, we have that $ex\pi(K), ex\tau(K) \in \Sigma_K$.
According to Choquet's integral representation theorem, there exists a σ-additive probability m_v on $ex(K)$ such that $\forall A \in Bor(\tau)$,

$$v(A) = \int_{ex(K)} h_{A|ex(K)}\ dm_v = \int_{ex(K)} u_{\mathcal{F}}(A)\ dm_v(u_{\mathcal{F}})$$

Let us define, for $A \in Bor(\tau)$

$$v_c(A) = \int_{ex\tau(K)} u_{\mathcal{F}}(A)\ dm_v(u_{\mathcal{F}}); v_p(A) = \int_{ex\pi(K)} u_{\mathcal{F}}(A)\ dm_v(u_{\mathcal{F}})$$

We prove that (v_c, v_p) is a decomposition we are looking for.
We first check that v_c, v_p are totally monotone. That v_c, v_p are monotone is straightforward since for $A, B \in Bor(\tau)$ with $A \subset B$ we have $h_{A|ex(K)} \leq h_{B|ex(K)}$ (since any filter game is monotone) and integration on $ex\tau(K), ex\pi(K)$ entails $v_c(A) \leq v_c(B)$ and $v_p(A) \leq v_p(B)$. Let $A_1, \ldots, A_K \in Bor(\tau)$, we have

$$h_{\cup_{k=1}^K A_k|ex(K)} \geq \sum_{\{I:\emptyset \neq I \subset \{1,\ldots,K\}\}} (-1)^{|I|+1} h_{\cap_{k\in I} A_k|ex(K)}$$

since any filter game is totally monotone. Integration on $ex\tau(K), ex\pi(K)$ entails

$$v_c(\cup_{k=1}^K A_k) \geq \sum_{\{I:\emptyset \neq I \subset \{1,\ldots,K\}\}} (-1)^{|I|+1} v_c(\cap_{k\in I} A_k)$$

and

$$v_p(\cup_{k=1}^K A_k) \geq \sum_{\{I:\emptyset \neq I \subset \{1,\ldots,K\}\}} (-1)^{|I|+1} v_p(\cap_{k\in I} A_k)$$

By the monotone convergence theorem we conclude that v_c is τ-continuous. And since $u_{\mathcal{F}|\mathcal{K}} = 0$ for pure filter games, integration on $ex\pi(K)$ insures that $v_{p|\mathcal{K}} = 0$, thus purity. It remains to prove the uniqueness properties of the decomposition.
For a totally monotone game $w \neq 0$, we associate a σ-additive measure on $ex(K)$ through

$$m_w = w(\Omega)\ m_{w/w(\Omega)}$$

We can notice that if w is a totally monotone game then w is τ-continuous if and only if $m_w(ex\pi(K)) = 0$.
(if) Assume $m_w(ex\pi(K)) = 0$. This part comes by the monotone convergence theorem. Let $\{A_n\}$ be a τ non-decreasing sequence converging to Ω. We have,

$$w(A_n) = \int_{ex(K)} u_{\mathcal{F}}(A_n)\ dm_w(u_{\mathcal{F}}) = \int_{ex\tau(K)} u_{\mathcal{F}}(A_n)\ dm_w(u_{\mathcal{F}})$$

$$\uparrow \int_{ex\tau(K)} u_{\mathcal{F}}(\Omega)\ dm_w(u_{\mathcal{F}}) = m_w(ex\tau(K)) = m_w(ex(K)) = w(\Omega)$$

(only if) We have from τ-continuity,

$$w(K_n) = m_w(\tilde{K_n}) \uparrow w(\Omega) = m_w(ex(K))$$

Now since $\cup_n \tilde{K_n} = ex\tau(K)$ and m_w is σ-additive this gives $m_w(ex\tau(K)) = m_w(ex(K))$ thus $m_w(ex\pi(K)) = 0$.
Now we obtain,

$$v_c(A) = m_v(\tilde{A} \cap ex\tau(K)) = m_v(\tilde{A} \cap (\cup_n \tilde{K_n})) = m_v(\cup_n(\tilde{A} \cap \tilde{K_n}))$$

$$= m_v(\cup_n(\widetilde{A\cap K_n})) = \lim_n m_v(\widetilde{A\cap K_n}) = \lim_n v(A\cap K_n) = \lim_n v_c(A\cap K_n)$$

Since $A\cap K_n \subset K_n \in \mathcal{K}$ and $v_p(K_n) = 0$.
Let (v_1, v_2) be another decomposition.
Since v_2 is pure then $v_{2|\mathcal{K}} = 0 (= v_{p|\mathcal{K}})$. Let $A \in Bor(\tau)$ such that $\overline{A} \in \mathcal{K}$ then

$$0 \leq v_2(A) \leq v_2(\overline{A}) = 0, 0 \leq v_p(A) \leq v_p(\overline{A}) = 0$$

and $v_1(A) = v_c(A)$.
If $v_1 = 0$ then $0 \leq v_c(K_n) \leq v_2(K_n) = 0$, thus by τ-continuity $v_c(\Omega) = 0$, so $v_c = 0 = v_1$ and $v_p = v_2$.
If $v_1 \neq 0$ then for $A \in Bor(\tau)$ we have,

$$v_1(A) = m_{v1}(\tilde{A}) = m_{v1}(\tilde{A}\cap ex\tau(K)) + m_{v1}(\tilde{A}\cap ex\pi(K)) = m_{v1}(\tilde{A}\cap ex\tau(K))$$

since $m_{v1}(ex\pi(K)) = 0$ as soon as v_1 is τ-continuous. And

$$m_{v1}(\tilde{A}\cap ex\tau(K)) = m_{v_1}(\tilde{A}\cap(\cup_n\tilde{K_n})) = m_{v_1}(\cup_n(\tilde{A}\cap\tilde{K_n}))$$

$$= m_{v_1}(\cup_n(\widetilde{A\cap K_n})) = \lim_n m_{v_1}(\widetilde{A\cap K_n})$$

$$= \lim_n v_1(A\cap K_n) = \lim_n v_c(A\cap K_n) = v_c(A)$$

since for all n, $A\cap K_n \subset K_n \in \mathcal{K}$ which entails $v_1(A\cap K_n) = v_c(A\cap K_n)$. So $v_1 = v_c$, thus $v_2 = v_p$. $\square$

A simple characterization of τ-continuous totally monotone games can be deduce,

Corollary 1 *Let v be a totally monotone game on $Bor(\tau)$. The following statements are equivalent,*
(i) v is τ-continuous,
(ii) $v(\Omega) = \sup\{v(K) : K \in \mathcal{K}\}$,
(iii) $v(\Omega) = \lim_n v(K_n)$.

Proof : $(i) \Rightarrow (iii)$. By definition.
$(iii) \Rightarrow (ii)$. Since $\{K_n\}_n \subset \mathcal{K}$, we have $v(\Omega) \geq \sup\{v(K) : K \in \mathcal{K}\} \geq \lim_n v(K_n) = v(\Omega)$.
$(ii) \Rightarrow (i)$. Let (v_c, v_p) be the Yosida-Hewitt decomposition of v. We have,

$$v(\Omega) = \sup_{\mathcal{K}} v(K) \leq \sup_{\mathcal{K}} v_c(K) + \sup_{\mathcal{K}} v_p(K) = \sup_{\mathcal{K}} v_c(K) \leq v_c(\Omega) \leq v(\Omega)$$

So $v_p(\Omega) = 0$, thus $v_p = 0$ and $v = v_c$. $\square$

We retrieve a Yosida-Hewitt decomposition theorem for measures and also for minitive measures,

Corollary 2 *Let v be a measure on $Bor(\tau)$ then there exists a unique pair of measures (v_c, v_p) with v_c τ-continuous and v_p pure such that $v = v_c + v_p$.*

Proof : See Corollary 1 in [3]. $\square$

Corollary 3 *Let v be a minitive measure on $Bor(\tau)$ then there exists a unique pair of minitive measures (v_c, v_p) with v_c τ-continuous and v_p pure such that $v = v_c + v_p$.*

Proof : See Property (iv) in [9]. $\square$

We finally give a simple sufficient condition on τ in order to obtain the previous results in [3].

Corollary 4 *Assume (Ω, τ) is discrete i.e. $\tau = cl(\tau)$. Let v be a totally monotone game then there exists a unique pair of totally monotone games (v_c, v_p) with v_c σ-continuous and v_p σ-pure* [5] *such that $v = v_c + v_p$.*

Proof : Notice that if (Ω, τ) is discrete and σ-compact then Ω is necessarily countable, since its compact sets are finite. Now since $\tau = \mathcal{P}(\Omega)$ then τ-continuity and σ-continuity coincide, thus their respective notion of purity. $\square$

4 Extension to Choquet functionals

A canonical way to extend our study of the Yosida-Hewitt decomposition of totally monotone games to functionals is to consider the *Choquet* integral ([4]). Let v be a totally monotone game on $Bor(\tau)$ and f be a non-negative bounded measurable i.e. $f \in B^+_\infty(\Omega, Bor(\tau))$ (B for short), the Choquet integral is given through

$$\int f \; dv = \int_0^\infty v(\{\omega : f(\omega) > t\}) \; dt$$

where the integral under consideration is a Riemann integral. An essential property of the Choquet is to be additive for *comonotonic* functions ([5]),

$$\int f + g \; dv = \int f \; dv + \int g \; dv$$

provided

$$\forall \omega, \omega' \in \Omega, (f(\omega) - f(\omega'))(g(\omega) - g(\omega')) \geq 0$$

A functional $I : B \longrightarrow \mathbb{R}$ is *totally monotone* if $f \geq g \Rightarrow I(f) \geq I(g)$ and

$$\forall n \geq 1, \forall f_1, \dots f_n \in B, I(\vee_1^n f_i) \geq \sum_{\{I:\emptyset \neq I \subset \{1,\dots,K\}\}} (-1)^{|I|+1} I(\wedge_{k \in I} f_k)$$

Where $\vee, \wedge$ stand for the usual Sup, Inf operators on functions.
From Schmeidler's ([11]) integral representation theorem we have a simple characterization of totally monotone functionals,

[5]In the sense of [3], i.e. $\forall \nu$ σ-continuous, $v_p \geq \nu \geq 0 \Rightarrow \nu = 0$

Theorem : *Let $I : B \longrightarrow \mathbb{R}$ be a functional. Define $v : Bor(\tau) \longrightarrow \mathbb{R}$ through $v(A) = I(1_A)$. Then v is totally monotone and $I = \int(.)\, dv$ if and only if I is totally monotone and comonotonic additive.*

Denote by USC_c (resp. LSC_c) the set of upper semi-continuous (resp. lower) non-negative functions with compact support i.e. $S(f) = \overline{\{\omega : f(\omega) > 0\}} \in \mathcal{K}$. We can characterize further the functionals associated to a τ-continuous totally monotone games,

Proposition 1 *Let I, v be associated as in Theorem 2. Then v is τ-continuous if and only if for all $\{f_n\}_n \in Bor(\tau), \forall \{g_n\}_n \in LSC_c$ if $f_n \leq g_n \leq f_{n+1}$ for all n and $\vee_n f_n = 1_\Omega$ then $I(f_n) \uparrow I(1_\Omega)$.*

Proof : (if) It suffices to check continuity of v at Ω. Take $f_n = 1_{A_n}, g_n = 1_{A^o_{n+1}}$ and $f = 1_\Omega$.
(only if) Assume v is τ-continuous or equivalently $\forall \{A_n\}_n \subset Bor(\tau), \forall \{B_n\}_n \subset \tau$ if $A_n \subset B_n$ and $\cup_n A_n = \Omega$ then $v(A_n) \uparrow v(\Omega)$.
Let $\{f_n\}_n \in Bor(\tau), \{g_n\}_n \in LSC_c$ such that $f_n \leq g_n \leq f_{n+1}$ for all n and $\vee_n f_n = 1_\Omega$. Let $t \in (0,1)$, put $A^t_n = \{f_n > t\} \in Bor(\tau)$, $B^t_n = \{g_n > t\} \in \tau$, then $\cup_n A^t_n = \Omega$. Since v is τ-continuous we have $v(A^t_n) \uparrow v(\Omega)$, and via the monontone convergence theorem we get

$$I(f_n) = \int_0^\infty v(A^t_n)\, dt = \int_{[0,1]} v(A^t_n)\, d\lambda(t) \uparrow \int_{[0,1]} v(\Omega)\, d\lambda(t) = v(\Omega) = I(1_\Omega)$$

where λ is the usual Lebesgue measure on $\mathbb{R}$. □

The definition of purity can be adapted in the functional setting. A totally monotone comonotone additive functional I is said to be *pure* if $\forall$ J τ-continuous totally monotone comonotone additive functional, $0 \leq J \leq I \Rightarrow J = 0$.

Similarly to the case of totally monotone game we obtain a characterization of purity in terms of upper semi-continuous functions with compact support,

Proposition 2 *Let I be a totally monotone functional comonotone additive then I is pure if and only if $I_{|USC_c} = 0$.*

Proof : (if) Let J be a totally monotone comonotone additive functional such that $0 \leq J \leq I$ and $\{g_n\}_n \in LSC_c$ with $g_n \leq g_{n+1}$ for all n and $\vee_n g_n = 1_\Omega$ (for instance $g_n = 1_{K^o_n}$). Since for all n we have $g_n \leq 1_{S(g_n)} \in USC_c$ we have $J(g_n) \leq I(g_n) \leq I(1_{S(g_n)}) = 0$, letting n go to ∞ entails $J(1_\Omega) = 0$. Thus by monotonicity, $J = 0$.
(only if) Let $v(A) = I(1_A)$ for all $A \in Bor(\tau)$. If I is pure, then by Proposition 1, v is pure in the sense of totally monotone games. Let $f \in USC_c$. We have $f \leq 1_{S(f)} \in USC_c$ so $I(f) \leq I(1_{S(f)}) = v(S(f)) = 0$ (by Lemma 1). □

A typical example of pure totally monotone functional comonotone additive is given by $\int(.)du_{co\mathcal{K}}$. This functional plays the rôle of a generalized *limes inferior* since from [8] we can get

$$\forall f \in B, \int f du_{co\mathcal{K}} = \liminf_{\mathcal{K},\supset \; K^c} f$$

which turns out to be equal in our setting to $\underline{lim} f := \mathrm{Sup}_n \mathrm{Inf}_{K_n^c} f$, since for all $K \in \mathcal{K}, \exists n / K \subset K_n$. In particular, for $f \in B$, f is *continuous at* ∞ i.e.

$$\exists l \in \mathbb{R}^+ \; /\forall \epsilon > 0, \exists n/ \forall \omega \notin K_n, |f(\omega) - l| < \epsilon$$

if and only if

$$\underline{lim} f = \overline{lim} f := \mathrm{Inf}_n \mathrm{Sup}_{K_n^c} f.$$

We have a similar Yosida-Hewitt decomposition for totally monotone comonotone additive functionals,

Theorem 2 *Let I be a totally monotone comonotone additive functional then there exists a unique pair of totally monotone comonotone additive functionals (I_c, I_p) with I_c τ-continuous and I_p pure such that $I = I_c + I_p$.*

Proof : Let $v(A) = I(1_A)$ for all $A \in Bor(\tau)$. Let (v_c, v_p) be the unique decomposition of v given by Theorem 1. Then, thanks to Propositions 1 and 2, $I_c = \int(.)\; dv_c$ and $I_p = \int(.)\; dv_p$ is the decomposition we are looking for. $\square$

5 Concluding comments

In this paper we prove a Yosida-Hewitt theorem for totally monotone games defined on a locally compact σ-compact topological space, which allows one to separate a τ-continuous component and a purely non τ-continuous part. A similar result is achieved for totally monotone comonotone additive functionals. Our intend for future researches is to explore the ability of our method, based on the Choquet integral representation theorem, to obtain a Yosida-Hewitt decomposition on measurable spaces.

Appendix

Lemma: Let v be monotone and monotone of order 2. Then v is τ-continuous if and only if v is *continuous at* Ω.

Proof : (only if). Is immediate.
(if). We first prove continuity from below. Let $\{A_n\}_n \subset Bor(\tau)$ be a non-decreasing sequence converging to $A \in cl(\tau)$. Put $B_n = A_n \cup A^c$ for all n. Since $A \in cl(\tau)$ we have,

$$B_n = A_n \cup A^c \subset A_{n+1}^o \cup A^c \subset (A_{n+1} \cup A^c)^o = B_{n+1}^o$$

so $\{B_n\}_n$ is a non-decreasing sequence converging to Ω. Now convexity entails, $v(\Omega) + v(A_n) \geq v(B_n) + v(A)$, thus letting n go to ∞, gives $\lim_n v(A_n) \geq v(A)$.

Let $\{A_n\}_n \subset Bor(\tau)$ be a non-increasing sequence converging to $A \in \tau$. Put $B_n = A \cup A_n^c$ for all n. Since for all n, $A_n \supset \overline{A_{n+1}} = ((A_{n+1}^c)^o)^c$ we have $A_n^c \subset (A_{n+1}^c)^o$, hence

$$B_n = A \cup A_n^c \subset A^o \cup (A_{n+1}^c)^o \subset (A \cup A_{n+1}^c)^o = B_{n+1}^o$$

Convexity entails, $v(\Omega) + v(A) \geq v(A_n) + v(B_n)$, thus letting n go to ∞, gives $v(A) \geq \lim_n v(A_n)$. □

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